MATH 2230 Complex Variables and Application
Suggested Solution to HW9

SEC. 66

9. Solution: $f(z)=\sin z^2 = \sum_{n=0}^{\infty} (-1)^n \frac{z^{4n+2}}{(2n+1)!} = z^2 - \frac{z^6}{3!} + \frac{z^{10}}{5!} - \cdots$

Since there are no terms of z^{4n} and z^{2n+1} in the Maclaurin series, we have $f^{(4n)}(0)=0$ and $f^{(2n+1)}(0)=0$ for each n.

10. (a) When $0<|z|<\infty$,

$$\frac{\sinh z}{z^2} = \sum_{n=0}^{\infty} \frac{z^{2n-1}}{(2n+1)!} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{z^{2n-1}}{(2n+1)!} = \frac{1}{z} + \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+3)!}$$

(b) When $0<|z|<\infty$,

$$\frac{\sin z^2}{z^4} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{4n-2}}{(2n+1)!} = \frac{1}{z^2} - \frac{z^2}{3!} + \frac{z^6}{5!} - \frac{z^{10}}{7!} + \cdots$$

11. proof: Since $0<|z|<4$, we have $\left|\frac{z}{4}\right|<1$.

Then $$\frac{1}{4z-z^2} = \frac{1}{4z}\cdot\frac{1}{1-\frac{z}{4}} = \frac{1}{4z}\sum_{n=0}^{\infty}\left(\frac{z}{4}\right)^n = \frac{1}{4z} + \sum_{n=0}^{\infty}\frac{z^n}{4^{n+2}}$$

SEC. 68

1. Solution: $f(z) = z^2\sin(z^{-2}) = z^2\sum_{n=0}^{\infty}(-1)^n \frac{z^{-4n-2}}{(2n+1)^2}$

$$= 1 + \sum_{n=1}^{\infty}(-1)^n/(2n+1)^2\cdot\frac{1}{z^{4n}} \qquad (0<|z|<\infty)$$

2. Solution: Since $1<|z|<\infty$, we have $\left|-\frac{1}{z}\right|<1$.

Then $$f(z) = \frac{1}{z}\cdot\frac{1}{1-(-\frac{1}{z})} = \frac{1}{z}\cdot\sum_{n=0}^{\infty}\left(-\frac{1}{z}\right)^n = \sum_{n=0}^{\infty}(-1)^n\frac{1}{z^{n+1}} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{z^n}$$

4. Solution: Let $f(z) = \frac{1}{z^2(1-z)} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{1-z}$.

Then $A=B=C=1$ and $f(z) = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{1-z}$.

When $0<|z|<1$,

$$f(z) = \frac{1}{z} + \frac{1}{z^2} + \sum_{n=0}^{\infty} z^n$$

When $1<|z|<\infty$, $\left|\frac{1}{z}\right|<1$,

$$f(z) = \frac{1}{z} + \frac{1}{z^2} + \left(-\frac{1}{z}\right)\cdot\frac{1}{(1-\frac{1}{z})} = \frac{1}{z} + \frac{1}{z^2} + \left(-\frac{1}{z}\right)\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n = -\sum_{n=3}^{\infty}\frac{1}{z^n}$$

5. When $|z|<1$, we have $\left|\frac{z}{2}\right|<1$

$$f(z) = -\frac{1}{1-z} + \frac{1}{2}\cdot\frac{1}{1-\frac{z}{2}} = -\sum_{n=0}^{\infty} z^n + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n = \sum_{n=0}^{\infty}\left(2^{-n-1}-1\right)z^n$$

When $1<|z|<2$, we have $\left|\frac{1}{z}\right|<1$ and $\left|\frac{z}{2}\right|<1$

$$f(z) = \frac{1}{z}\cdot\frac{1}{1-\frac{1}{z}} + \frac{1}{2}\cdot\frac{1}{1-\frac{z}{2}} = \frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$

$$= \sum_{n=1}^{\infty}\frac{1}{z^n} + \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}$$

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When $2<|z|<\infty$,

$$f(z)=\frac{1}{z}\cdot\frac{1}{1-\frac{1}{z}}-\frac{1}{z}\cdot\frac{1}{1-\frac{2}{z}}=\frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n}-\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n=\sum_{n=1}^{\infty}\frac{1}{z^n}\left(1-2^{n-1}\right)$$

6. proof: $\frac{z}{(z-1)(z-3)}=\frac{-\frac{1}{2}}{z-1}+\frac{\frac{3}{2}}{z-3}$

$$=-\frac{1}{2(z-1)}+\frac{3}{2}\cdot\frac{1}{(z-1)-2}=-\frac{1}{2(z-1)}-\frac{3}{4}\cdot\frac{1}{1-\frac{z-1}{2}}$$

$$=-\frac{1}{2(z-1)}-\frac{3}{4}\sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^n=-3\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^{n+2}}-\frac{1}{2(z-1)}$$

7. (a) When $|a|<|z|<\infty$, we have $\left|\frac{a}{z}\right|<1$

$$\frac{a}{z-a}=\frac{a}{z}\cdot\frac{1}{1-\frac{a}{z}}=\frac{a}{z}\sum_{n=0}^{\infty}\left(\frac{a}{z}\right)^n=\sum_{n=1}^{\infty}\frac{a^n}{z^n}$$

(b) When $z=e^{i\theta}$, we have

$$\frac{a}{e^{i\theta}-a}=\frac{a}{\cos\theta-a+i\sin\theta}=\frac{a(\cos\theta-a)-ia\sin\theta}{(\cos\theta-a)^2+\sin^2\theta}=\frac{a\cos\theta-a^2-ia\sin\theta}{1-2a\cos\theta+a^2}$$

and $\sum_{n=1}^{\infty}\frac{a^n}{e^{in\theta}}=\sum_{n=1}^{\infty}(\cos n\theta-i\sin n\theta)\,a^n=\sum_{n=1}^{\infty}\cos n\theta\, a^n-i\sum_{n=1}^{\infty}\sin n\theta\, a^n$

Thus, $\sum_{n=1}^{\infty}a^n\cos n\theta=\frac{a\cos\theta-a^2}{1-2a\cos\theta+a^2}$

$$\sum_{n=1}^{\infty}a^n\sin n\theta=\frac{a\sin\theta}{1-2a\cos\theta+a^2}$$